The distance geometry of music

Erik D. Demaine a, Francisco Gomez-Martin b, Henk Meijer c, David Rappaport c, Perouz Taslakian d,*, Godfried T. Toussaint d, e, 2, Terry Winograd f, David R. Wood g

a Computer Science and Artificial Intelligence Laboratory, Massachusetts Institute of Technology, Cambridge, MA, USA
b Departamento de Matemática Aplicada, Universidad Politécnica de Madrid, Madrid, Spain
c School of Computing, Queen’s University, Kingston, Ontario, Canada
d School of Computer Science, McGill University, Montréal, Québec, Canada
e Centre for Interdisciplinary Research in Music Media and Technology, The Schulich School of Music, McGill University
f Department of Computer Science, Stanford University, Stanford, CA, USA
g Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Barcelona, Spain

ARTICLE INFO

Article history:
Received 4 May 2006
Received in revised form 18 April 2008
Accepted 19 April 2008
Available online 24 December 2008
Communicated by P. Bose and A. Mukhopadhyay

Keywords:
Maximally even
Euclidean rhythms
Deep rhythms
Euclidean algorithm
Generated rhythms

ABSTRACT

We demonstrate relationships between the classic Euclidean algorithm and many other fields of study, particularly in the context of music and distance geometry. Specifically, we show how the structure of the Euclidean algorithm defines a family of rhythms which encompass over forty timelines (ostinatos) from traditional world music. We prove that these Euclidean rhythms have the mathematical property that their onset patterns are distributed as evenly as possible: they maximize the sum of the Euclidean distances between all pairs of onsets, viewing onsets as points on a circle. Indeed, Euclidean rhythms are the unique rhythms that maximize this notion of evenness. We also show that essentially all Euclidean rhythms are deep: each distinct distance between onsets occurs with a unique multiplicity, and these multiplicities form an interval 1, 2, …, k − 1. Finally, we characterize all deep rhythms, showing that they form a subclass of generated rhythms, which in turn proves a useful property called shelling. All of our results for musical rhythms apply equally well to musical scales. In addition, many of the problems we explore are interesting in their own right as distance geometry problems on the circle; some of the same problems were explored by Erdős in the plane.

© 2009 Elsevier B.V. All rights reserved.

1. Introduction

Polygons on a circular lattice, African bell rhythms [86], musical scales [23], spallation neutron source accelerators in nuclear physics [16], linear sequences in mathematics [63], mechanical words and stringology in computer science [62], drawing digital straight lines in computer graphics [59], calculating leap years in calendar design [7,51], and an ancient algorithm for computing the greatest common divisor of two numbers, originally described by Euclid [39,44]—what do these disparate concepts all have in common? The short answer is, “patterns distributed as evenly as possible”. For the long answer, please read on.

* Corresponding author.
E-mail addresses: edemaine@mit.edu (E.D. Demaine), fmartin@eui.upm.es (F. Gomez-Martin), henk@cs.queensu.ca (H. Meijer), daver@cs.queensu.ca (D. Rappaport), perouz@cs.mcgill.ca (P. Taslakian), godfried@cs.mcgill.ca (G.T. Toussaint), winograd@cs.stanford.edu (T. Winograd), david.wood@upc.edu (D.R. Wood).

1 Supported by NSERC.
2 Supported by FQRNT and NSERC.
0925-7721/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.comgeo.2008.04.005
Mathematics and music have been intimately intertwined since over 2500 years ago when the famous Greek mathematician, Pythagoras of Samos (circa 500 B.C.), developed a theory of consonants based on ratios of small integers [8,10]. Most of this interaction between the two fields, however, has been in the domain of pitch and scales. For some historical snapshots of this interaction, we refer the reader to H.S.M. Coxeter’s delightful account [32]. In music theory, much attention has been devoted to the study of intervals used in pitch scales [43], but relatively little work has been devoted to the analysis of time duration intervals of rhythm. Some notable recent exceptions are the books by Simha Arom [3], Justin London [61] and Christopher Hasty [49].

In this paper, we study various mathematical properties of musical rhythms and scales that are all, at some level, connected to an algorithm of another famous ancient Greek mathematician, Euclid of Alexandria (circa 300 B.C.). We begin (in Section 2) by showing several mathematical connections between musical rhythms and scales, the work of Euclid, and other areas of knowledge such as nuclear physics, calendar design, mathematical sequences, and computer science. In particular, we define the notion of Euclidean rhythms, generated by an algorithm similar to Euclid’s. Then, in the more technical part of the paper (Sections 3–5.2), we study two important properties of rhythms and scales, called evenness and deepness, and show how these properties relate to the work of Euclid.

The Euclidean algorithm has been connected to music theory previously by Viggo Brun [20]. Brun used Euclidean algorithms to calculate the lengths of strings in musical instruments between two lengths \(l\) and \(2l\), so that all pairs of adjacent strings have the same length ratios. In contrast, we relate the Euclidean algorithm to rhythms and scales in world music.

Musical rhythms and scales can both be seen as two-way infinite binary sequences [85]. In a rhythm, each bit represents one unit of time called a pulse (for example, the length of a sixteenth note), a one bit represents a played note or onset (for example, a sixteenth note), and a zero bit represents a silence (for example, a sixteenth rest). In a scale, each bit represents a pitch (spaced uniformly in log-frequency space), and zero or one represents whether the pitch is absent or present in the scale. Here we suppose that all time intervals between onsets in a rhythm are multiples of a fixed time unit, and that all tone intervals between pitches in a scale are multiples of a fixed tonal unit (in logarithm of frequency).

The time dimension of rhythms and the pitch dimension of scales have an intrinsically cyclic nature, cycling every measure and every octave, respectively. In this paper, we consider rhythms and scales that match this cyclic nature of the underlying space. In the case of rhythms, such cyclic rhythms are also called timelines, rhythmic phrases or patterns that are repeated throughout a piece; in the remainder of the paper, we use the term “rhythm” to mean “timeline”. The infinite bit sequence representation of a cyclic rhythm or scale is just a cyclic repetition of some \(n\)-bit string, corresponding to the timespan of a single measure or the log-frequency span of a single octave. To properly represent the cyclic nature of this string, we imagine assigning the bits to \(n\) points equally spaced around a circle of circumference \(n\) [64]. A rhythm or scale can therefore be represented as a subset of these \(n\) points. We use \(k\) to denote the size of this subset; that is, \(k\) is the number of onsets in a rhythm or pitches in a scale. For uniformity, the terminology in the remainder of this paper speaks primarily about rhythms, but the notions and results apply equally well to scales.

In this paper, we use four representations of rhythms of timespan \(n\). The first representation is the commonly used box-like representation, also known as the Time Unit Box System (TUBS), which is a sequence of \(n\) ‘×’s and ‘.’s where ‘×’ represents an onset and ‘.’ represents a silence (a zero bit) [85]. This notation was used and taught in the West by Philip Harland at the University of California, Los Angeles, in 1962, and it was made popular in the field of ethnomusicology by James Koetting [58]. However, such box notation has been used in Korea for hundreds of years [50]. The second representation of rhythms and scales we use is the clockwise distance sequence, which is a sequence of integers that sum up to \(n\) and represent the lengths of the intervals between consecutive pairs of onsets, measuring clockwise arc-lengths or distances around the circle of circumference \(n\). The third representation of rhythms and scales writes the onsets as a subset of the set of all pulses, numbered 0, 1, \ldots, \(n - 1\), with a subscript of \(n\) on the right-hand side of the subset to denote the timespan. Clough and Douthett [22] use this notation to represent scales. For example, the Cuban clave Son rhythm can be represented as \([× · · × · · × · · × · · × · × · · × · × · · ]\) in box-like notation, \((3, 3, 4, 2, 4)\) in clockwise distance sequence notation, and \((0, 3, 6, 10, 12)\) in subset notation. Finally, the fourth representation is a graphical clock diagram [85], such as Fig. 1, in which the zero label denotes the start of the rhythm and time flows in a clockwise direction. In such clock diagrams we usually connect adjacent onsets by line segments, forming a polygon. We consider two rhythms distinct if their sequence of zeros and ones differ, starting from the first bit in any of the described representations. We assume that two rhythms that do not have the same sequence are different. However, if a rhythm is a rotated version of another, we say that they are instances of the same necklace. Thus a rhythm necklace is a clockwise distance sequence that disregards the starting point in the cycle. Note that the clockwise distance sequence notation requires that the rhythm starts with an onset, so it cannot be used to represent all rhythms; it is most useful for our analysis of necklaces.

Even rhythms. Consider the following three 12/8-time rhythms expressed in box-like notation: \([× · × · × · × · × · × · × · ]\), \([× · × · × · × · × · × · × · ]\), and \([× · × · × · × · × · × · × · ]\). The first rhythm contains beats that are distributed perfectly (well spaced). Such rhythms are found throughout the world, and are most easily identified and incorporated in music and dance. However, in many cultures where rhythm is more highly developed, rhythms are preferred that are not perfectly even. It is intuitively clear that the first rhythm is more even (well spaced) than the second rhythm, and that the second rhythm is more even than the third rhythm. In fact, the second rhythm is the internationally most well known of all African timelines. It is traditionally played on an iron bell, and is known on the world scene mainly by its Cuban name Bembé [86]. Traditional rhythms tend to exhibit such properties of evenness to some degree.
Why do many traditional rhythms display such evenness? Many are timelines (also sometimes called claves), that is, rhythms repeated throughout a piece that serve as a rhythmic reference point [67,91]. Often these claves have a call-and-response structure, meaning that the pattern is divided into two parts: the first poses a rhythmic question, usually by creating rhythmic tension, and the second part answers this question by releasing that tension. A good example of this structure is the popular clave Son [×···×···×···×···]. This clave creates such tension through syncopation, which can be found between the second and third onsets as well as between the third and fourth onsets. The latter is weak syncopation because the strong beat at position 8 lies half-way between the third and fourth onsets. (The strong beats of the underlying 4/4 meter (beat) occur at positions 0, 4, 8, and 12.) On the other hand, the former syncopation is strong because the strong beat at position 4 is closer to the second onset than to the third onset [47]. Claves played with instruments that produce unsustained notes often use syncopation and accentuation to bring about rhythmic tension. Many clave rhythms create syncopation by evenly distributing onsets in contradiction with the pulses of the underlying meter. For example, in the clave Son, the first three onsets are equally spaced at the distance of three sixteenth pulses, which forms a contradiction because 3 does not divide 16. Then, the response of the clave answers with an offbeat onset, followed by an onset on the fourth strong beat of a 4/4 meter, releasing that rhythmic tension.

On the other hand, a rhythm that is too even, such as the example [×·×·×·×·×·×·], is less interesting from a syncopation point of view. Indeed, in the most interesting rhythms with $k$ onsets and timespan $n$, $k$ and $n$ are relatively prime (have no common divisor larger than 1). This property is natural because the rhythmic contradiction is easier to obtain if the onsets do not coincide with the strong beats of the meter. Also, we find that many claves have an onset on the last strong beat of the meter, as does the clave Son. This is a natural way to respond in the call-and-response structure. A different case is that of the Bossa-Nova clave [×···×···×···×···]. This clave tries to break the feeling of the pulse and, although it is very even, it produces a cycle that perceptually does not coincide with the beginning of the meter.

This prevalence of evenness in world rhythms has led to the study of mathematical measures of evenness in the new field of mathematical ethnomusicology [26,88,89], where they may help to identify, if not explain, cultural preferences of rhythms in traditional music. Furthermore, evenness in musical chords plays a significant role in the efficacy of voice leading as discussed in the work of Tymoczko [52,90].

The notion of maximally even sets with respect to scales represented on a circle was introduced by Clough and Douthett [22]. According to Block and Douthett [12], Douthett and Entringer went further by constructing several mathematical

![Fig. 1. The six fundamental African and Latin American rhythms which all have equal sum of pairwise geodesic distances; yet intuitively, the Bossa-Nova rhythm is more “even” than the rest.](image-url)
measures of the amount of evenness contained in a scale; see [12, page 40]. One of their evenness measures simply sums the interval arc-lengths (geodesics along the circle) between all pairs of onsets (or more precisely, onset points). This measure differentiates between rhythms that differ widely from each other. For example, the two four-onset rhythms \([\times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \time
Then, through an unexpected connection to timing systems in neutron accelerators, we see how the same type of algorithm

2. Euclid and evenness in various disciplines

In this section, we first describe Euclid’s classic algorithm for computing the greatest common divisor of two integers. Then, through an unexpected connection to timing systems in neutron accelerators, we see how the same type of algorithm
can be used as an approach to maximizing “evenness” in a binary string with a specified number of zeroes and ones. This algorithm defines an important family of rhythms, called Euclidean rhythms, which we show appear throughout world music. Finally, we see how similar ideas have been used in algorithms for drawing digital straight lines and in combinatorial strings called Euclidean strings.

2.1. The Euclidean algorithm for greatest common divisors

The Euclidean algorithm for computing the greatest common divisor of two integers is one of the oldest known algorithms (circa 300 B.C.). It was first described by Euclid in Proposition 2 of Book VII of *Elements* [39,44]. Indeed, Donald Knuth [57] calls this algorithm the “granddaddy of all algorithms, because it is the oldest nontrivial algorithm that has survived to the present day”.

The idea of the algorithm is simple: repeatedly replace the larger of the two numbers by their difference until both are equal. This final number is then the greatest common divisor. For example, consider the numbers 5 and 13. First, 13 − 5 = 8; then 8 − 5 = 3; next 5 − 3 = 2; then 3 − 2 = 1; and finally 2 − 1 = 1. Therefore, the greatest common divisor of 5 and 13 is 1; in other words, 5 and 13 are relatively prime.

The algorithm can also be described succinctly in a recursive manner as follows [29]. Let $k$ and $n$ be the input integers with $k < n$.

**Algorithm.** Euclid$(k, n)$

1. if $k = 0$ then return $n$
2. else return Euclid$(n \mod k, k)$

Running this algorithm with $k = 5$ and $n = 13$, we obtain Euclid$(5, 13) = \text{Euclid}(3, 5) = \text{Euclid}(2, 3) = \text{Euclid}(1, 2) = \text{Euclid}(0, 1) = 1$. Note that this division version of Euclid’s algorithm skips one of the steps $(5, 8)$ made by the original subtraction version.

2.2. Evenness and timing systems in neutron accelerators

One of our main musical motivations is to find rhythms with a specified timespan and number of onsets that maximize evenness. Bjorklund [15,16] was faced with a similar problem of maximizing evenness, but in a different context: the operation of components such as high-voltage power supplies of spallation neutron source (SNS) accelerators used in nuclear physics. In this setting, a timing system controls a collection of gates over a time window divided into $n$ equal-length intervals. (In the case of SNS, each interval is 10 seconds.) The timing system can send signals to enable a gate during any desired subset of the $n$ intervals. For a given number $n$ of time intervals, and another given number $k < n$ of signals, the problem is to distribute the pulses as evenly as possible among these $n$ intervals. Bjorklund [16] represents this problem as a binary sequence of $k$ ones and $n − k$ zeroes, where each bit represents a time interval and the ones represent the times at which the timing system sends a signal. The problem then reduces to the following: construct a binary sequence of $n$ bits with $k$ ones such that the $k$ ones are distributed as evenly as possible among the $(n − k)$ zeroes.

One simple case is when $k$ evenly divides $n$ (without remainder), in which case we should place ones every $n/k$ bits. For example, if $n = 16$ and $k = 4$, then the solution is $[1000100010001000]$. This case corresponds to $n$ and $k$ having a common divisor of $k$. More generally, if the greatest common divisor between $n$ and $k$ is $g$, then we would expect the solution to decompose into $g$ repetitions of a sequence of $n/g$ bits. Intuitively, a string of maximum evenness should have this kind of symmetry, in which it decomposes into more than one repetition, whenever such symmetry is possible. This connection to greatest common divisors suggests that a rhythm of maximum evenness might be computed using an algorithm like Euclid’s. Indeed, Bjorklund’s algorithm closely mimics the structure Euclid’s algorithm, although this connection has never been mentioned before.

We describe Bjorklund’s algorithm by using one of his examples. Consider a sequence with $n = 13$ and $k = 5$. Because $13 − 5 = 8$, we start by considering a sequence consisting of 5 ones followed by 8 zeroes which should be thought of as 13 sequences of one bit each:

$$[1][1][1][1][1][0][0][0][0][0][0][0][0]$$

If there is more than one zero the algorithm moves zeroes in stages. We begin by taking zeroes one at a time (from right to left), placing a zero after each one (from left to right), to produce five sequences of two bits each, with three zeroes remaining:

$$[10][10][10][10][10][0][0][0]$$

Next we distribute the three remaining zeroes in a similar manner, by placing a [0] sequence after each [10] sequence:

$$[100][100][100][10][10]$$
Now we have three sequences of three bits each, and a remainder of two sequences of two bits each. Therefore we continue in the same manner, by placing a [10] sequence after each [100] sequence:

\[ [10010] [10010] [100] \]

The process stops when the remainder consists of only one sequence (in this case the sequence [100]), or we run out of zeroes (there is no remainder). The final sequence is thus the concatenation of [10010], [10010], and [100]:

\[ [10010100100] \]

We could proceed further in this process by inserting [100] into [10010] [10010]. However, Bjorklund argues that, because the sequence is cyclic, it does not matter (hence his stopping rule). For the same reason, if the initial sequence has a group of ones followed by only one zero, the zero is considered as a remainder consisting of one sequence of one bit, and hence nothing is done. Bjorklund [16] shows that the final sequence may be computed from the initial sequence using \( O(n) \) arithmetic operations in the worst case.

A more convenient and visually appealing way to implement this algorithm by hand is to perform the sequence of insertions in a vertical manner as follows. First take five zeroes from the right and place them under the five ones on the left:

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}
\]

Then move the three remaining zeroes in a similar manner:

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0
\end{array}
\]

Next place the two remainder columns on the right under the two leftmost columns:

\[
\begin{array}{ccc}
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 \\
1 & 1 \\
0 & 0
\end{array}
\]

Here the process stops because the remainder consists of only one column. The final sequence is obtained by concatenating the three columns from left to right:

\[ 1 & 0 & 1 & 0 & 1 & 0 & 0 \]

Bjorklund’s algorithm applied to a string of \( n \) bits consisting of \( k \) ones and \( n-k \) zeros has the same structure as running Euclid \((k,n)\). Indeed, Bjorklund’s algorithm uses the repeated subtraction form of division, just as Euclid did in his Elements [39]. It is also well known that applying the algorithm Euclid \((k,n)\) to two \( O(n) \) bit numbers (binary sequences of length \( n \)) causes it to perform \( O(n) \) arithmetic operations in the worst case [29].

### 2.3. Euclidean rhythms

The binary sequences generated by Bjorklund’s algorithm, as described in the preceding, may be considered as one family of rhythms. Furthermore, because Bjorklund’s algorithm is a way of visualizing the repeated-subtraction version of the Euclidean algorithm, we call these rhythms Euclidean rhythms. We denote the Euclidean rhythm by \( E(k,n) \), where \( k \) is the number of ones (onsets) and \( n \) (the number of pulses) is the length of the sequence (zeros plus ones). For example, \( E(5,13) = [10010100100100] \). The zero-one notation is not ideal for representing binary rhythms because it is difficult to visualize the locations of the onsets as well as the duration of the inter-onset intervals. In the more iconic box notation, the preceding rhythm is written as \( E(5,13) = [\times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \time
African traditional music. For example, it is played on the atoke bell in the Sohu, an Ewe dance from Ghana [54]. The tresillo can also be recognized as the first bar (first eight pulses) of the ubiquitous two-bar clave Son shown in Fig. 1(b).

In the two examples $E(5, 13)$ and $E(3, 8)$, there are fewer ones than zeros. If instead there are more ones than zeros, Bjorklund’s algorithm yields the following steps with, for example, $k = 5$ and $n = 8$:

$$
\begin{align*}
1 & 1 1 1 1 0 0 0 \\
10 & 10 10 10 & 11 \\
101 & 101 & 10 \\
10110110 & & \\
10110110 & & \\
\end{align*}
$$

The resulting Euclidean rhythm is $E(5, 8) = [\ldots \times \times \times \times \times \times \times \ldots]$. Fig. 3(b) shows a clock diagram for this rhythm. It is another famous rhythm on the world scene. In Cuba, it goes by the name of the cinquillo and it is intimately related to the tresillo [42]. It has been used in jazz throughout the 20th century [72], and in rockabilly music. For example, it is the hand-clapping pattern in Elvis Presley’s Hound Dog [19]. The cinquillo pattern is also widely used in West African traditional music [71,85], as well as Egyptian [48] and Korean [50] music.

We show in this paper that Euclidean rhythms have two important properties: they maximize evenness and they are deep. The evenness property should come as no surprise, given how we designed the family of rhythms. To give some feeling for the deepness property, we consider the two examples in Fig. 3, which have been labeled with the distances between all pairs of onsets, measured as arc-lengths. The tresillo in Fig. 3(a) has one occurrence of distance 2 and two occurrences of distance 3. The cinquillo in Fig. 3(b) contains one occurrence of distance 4, two occurrences of distance 1, three occurrences of distance 2, and four occurrences of distance 3. Thus, every distance has a unique multiplicity, making these rhythms Erdős-deep.

2.5. Aksak rhythms

Euclidean rhythms are closely related to a family of rhythms known as aksak rhythms, which have been studied from the combinatorial point of view for some time [4,17,28]. Béla Bartók [9] and Constantin Brăiloiu [17], respectively, have used
the terms Bulgarian rhythm and aksak to refer to those meters that use units of durations 2 and 3, and no other durations. Furthermore, the rhythm or meter must contain at least one duration of length 2 and at least one duration of length 3. Arom [4] refers to these durations as binary cells and ternary cells, respectively.

Arom [4] generated an inventory of all the theoretically possible aksak rhythms for values of \( n \) ranging from 5 to 29, as well as a list of those that are actually used in traditional world music. He also proposed a classification of these rhythms into several classes, based on structural and numeric properties. Three of his classes are considered here:

1. An aksak rhythm is authentic if \( n \) is a prime number.
2. An aksak rhythm is quasi-aksak if \( n \) is an odd number that is not prime.
3. An aksak rhythm is pseudo-aksak if \( n \) is an even number.

A quick perusal of the Euclidean rhythms listed in the preceding reveals that aksak rhythms are well represented. Indeed, all three of Arom’s classes (authentic, quasi-aksak, and pseudo-aksak) make their appearance. There is a simple characterization of those rhythms that are aksak. From the iterative subtraction algorithm of Bjorklund it follows that if \( n = 2k \) all cells are binary (duration 2). Similarly, if \( n = 3k \) all cells are ternary (duration 3). Therefore, to ensure that the Euclidean rhythm contains both binary and ternary cells, and no other durations, it follows that \( n \) must be between 2k and 3k.

Of course, not all aksak rhythms are Euclidean. Consider the Bulgarian rhythm with interval sequence \((3322)\) [4], which is also the metric pattern of Indian Lady by Don Ellis [55]. Here \( k = 4 \) and \( n = 10 \), and \( E(4, 10) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot] = (3232) \) (classical music, Bulgaria, Northern India, Serbia).

The following Euclidean rhythms are authentic aksak:

\[
E(2, 5) = [\times \cdot \cdot \cdot] = (23) \quad \text{(classical music, jazz, Greece, Macedonia, Namibia, Persia, Rwanda)}.
\]
\[
E(3, 7) = [\times \times \times \times \times \times \times] = (223) \quad \text{(Bulgaria, Greece, Sudan, Turkestan)}.
\]
\[
E(4, 11) = [\times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \time...
Fig. 5. The shaded pixels form a digital straight line determined by the points \( p \) and \( q \).

\[
E(7, 16) = [\ldots \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \time...
convention as to how to choose a canonical rhythm that represents the necklace. Whether there is anything musically meaningful about these conventions is left to ethnomusicologists to decide.

Let \( P = (p_0, p_1, \ldots, p_{n-1}) \) denote a string of nonnegative integers. Let \( \rho(P) \) denote the right rotation of \( P \) by one position; that is, \( \rho(P) = (p_{n-1}, p_0, p_1, \ldots, p_{n-2}) \). Let \( \rho^d(P) \) denote the right rotation of \( P \) by \( d \) positions. If \( P \) is considered as a cyclic string, a right rotation corresponds to a clockwise rotation. Fig. 6 illustrates the \( \rho(P) \) operator with \( P \) equal to the Bembé bell-pattern of West Africa [86]. Fig. 6(a) shows the Bembé bell-pattern, Fig. 6(b) shows \( \rho(P) \), which is a hand-clapping pattern from West Africa [70], and Fig. 6(c) shows \( \rho^3(P) \), which is the Tambú rhythm of Curaçao [77].

Ellis et al. [38] define a string \( P = (p_0, p_1, \ldots, p_{n-1}) \) to be Euclidean if incrementing \( p_0 \) by 1 and decrementing \( p_{n-1} \) by 1 yields a new string \( \tau(P) \) that is the rotation of \( P \). In other words, \( P \) and \( \tau(P) \) are instances of the same necklace. Therefore, if we represent rhythms as binary sequences, Euclidean rhythms cannot be Euclidean strings because all Euclidean rhythms begin with a ‘one’. Increasing \( p_0 \) by one makes it a ‘two’, which is not a binary string. Therefore, to explore the relationship between Euclidean strings and Euclidean rhythms, we will represent rhythms by their clockwise distance sequences, which are also strings of nonnegative integers. An example, consider \( E(4, 9) = [\times \times \times \times] \) (2223). Now \( \tau(2223) = (3222) \), which is a rotation of \( E(4, 9) \), and thus \( (2222) \) is a Euclidean string. Indeed, for \( P = E(4, 9), \tau(P) = \rho^3(P) \). As a second example, consider the West African clapping-pattern shown in Fig. 6(b) given by \( P = (1212221) \). We have that \( \tau(P) = (2221221) = \rho^5(P) \), the pattern shown in Fig. 6(c), which also happens to be the mirror image of \( P \) about the \( (0, 6) \) axis. Therefore \( P \) is a Euclidean string. However, note that \( P \) is not a Euclidean rhythm. Nevertheless, \( P \) is a rotation of the Euclidean rhythm \( E(7, 12) = (2122122) \).

Ellis et al. [38] have many beautiful results about Euclidean strings. They show that Euclidean strings exist if, and only if, \( n \) and \( (p_0 + p_1 + \cdots + p_{n-1}) \) are relatively prime numbers, and that when they exist they are unique. They also show how to construct Euclidean strings using an algorithm that has the same structure as the Euclidean algorithm. In addition they relate Euclidean strings to many other families of sequences studied in the combinatorics of words [5,62].

Let \( R(P) \) denote the reversal (or mirror image) of \( P \); that is, \( R(P) = (p_{n-1}, p_{n-2}, \ldots, p_1, p_0) \). Now we may determine which of the Euclidean rhythms used in world music listed in the preceding, are Euclidean strings or reverse Euclidean strings. The length of a Euclidean string is defined as the number of integers it has. This translates in the rhythm domain to the number of onsets a rhythm contains. Furthermore, strings of length one are Euclidean strings, trivially. Therefore all the trivial Euclidean rhythms with only one onset, such as \( E(1, 2) = [\times \times] = (2) \), \( E(1, 3) = [\times \times \times] = (3) \), and \( E(1, 4) = [\times \times \times \times] = (4) \), etc., are both Euclidean strings as well as reverse Euclidean strings. In the lists that follow the Euclidean rhythms are shown in their box-notation format as well as in the clockwise distance sequence representation. The styles of music that use these rhythms is also included. Finally, if only a rotated version of the Euclidean rhythm is played, then it is still included in the list but referred to as a necklace.

The following Euclidean rhythms are Euclidean strings:

- \( E(2, 3) = [\times \times \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \·...
Strings (group three) fall into two categories: those consisting of clockwise distances 1 and 2, and those consisting of

The following Euclidean rhythms are neither Euclidean nor reverse Euclidean strings:

The Euclidean rhythms that appear in classical music and jazz are also Euclidean strings (the first group). Furthermore, this group is not popular in African music. The Euclidean rhythms that are neither Euclidean strings nor reverse Euclidean strings (group three) fall into two categories: those consisting of clockwise distances 1 and 2, and those consisting of
clockwise distances 2 and 3. The latter group is used only in Bulgaria, and the former is used in Africa. Finally, the Euclidean rhythms that are reverse Euclidean strings (the second group) appear to have a much wider use. Finding musical explanations for these mathematical properties raises interesting ethnomusicalogical questions.

The Euclidean strings defined in [38] determine another family of rhythms, many of which are also used in world music but are not necessarily Euclidean rhythms. For example, (1221222) is an Afro–Cuban bell pattern. Therefore it would be interesting to explore empirically the relation between Euclidean strings and world music rhythms, and to determine formally the exact mathematical relation between Euclidean rhythms and Euclidean strings.

3. Definitions and notation

Before we begin the more technical part of the paper, we need to define some precise mathematical notation for describing rhythms.

Let $\mathbb{Z}^+$ denote the set of positive integers. For $k, n \in \mathbb{Z}^+$, let $\gcd(k, n)$ denote the greatest common divisor of $k$ and $n$. If $\gcd(k, n) = 1$, we call $k$ and $n$ relatively prime. For integers $a < b$, let $[a, b] = \{a, a + 1, a + 2, \ldots, b\}$.

Let $C$ be a circle in the plane, and consider any two points $x, y$ on $C$. The chordal distance between $x$ and $y$, denoted by $d(x, y)$, is the length of the line segment $\overline{xy}$; that is, $d(x, y)$ is the Euclidean distance between $x$ and $y$. The clockwise distance from $x$ to $y$, or of the ordered pair $(x, y)$, is the length of the clockwise arc of $C$ from $x$ to $y$, and is denoted by $\overset{\leftarrow}{d}(x, y)$. Finally, the geodesic distance between $x$ and $y$, denoted by $\overset{\rightarrow}{d}(x, y)$, is the length of the shortest arc of $C$ between $x$ and $y$; that is, $\overset{\rightarrow}{d}(x, y) = \min\{d(x, y), \overset{\leftarrow}{d}(y, x)\}$.

A rhythm of timespan $n$ is a subset of $\{0, 1, \ldots, n - 1\}$, representing the set of pulses that are onsets in each repetition. For clarity, we write the timespan $n$ as a subscript after the subset: $[\ldots]_n$. Geometrically, if we locate $n$ equally spaced points clockwise around a circle $C_n$ of circumference $n$, then we can view a rhythm of timespan $n$ as a subset of these $n$ points. We consider an element of $\mathbb{Z}_n$ to simultaneously be a point on the circle and an integer in $\{0, 1, \ldots, n - 1\}$.

The rotation of a rhythm $R$ of timespan $n$ by an integer $\Delta \geq 0$ is the rhythm $[i + \Delta \mod n : i \in R]_n$ of the same timespan $n$. The scaling of a rhythm $R$ of timespan $n$ by an integer $\alpha \geq 1$ is the rhythm $[\alpha i : i \in R]_{\alpha n}$ of timespan $\alpha n$.

Let $R = [r_0, r_1, \ldots, r_{k-1}]_n$ be a rhythm of timespan $n$ with $k$ onsets sorted in clockwise order. Throughout this paper, an onset $r_i$ will mean $(r_i \mod k) \mod n$. Observe that the clockwise distance $\overset{\leftarrow}{d}(r_i, r_j) = (r_j - r_i) \mod n$ is the number of points on $C_n$ that are contained in the clockwise arc $(r_i, r_j)$ and is also known as the chromatic length [22].

The geodesic distance multiset of a rhythm $R$ is the multiset of all nonzero pairwise geodesic distances; that is, it is the multiset $\{\overset{\rightarrow}{d}(r_i, r_j) : r_i, r_j \in R, r_i \neq r_j\}$. The geodesic distance multiset has cardinality $\left\lfloor \frac{n}{2} \right\rfloor$. The multiplicity of a distance $d$ is the number of occurrences of $d$ in the geodesic distance multiset.

A rhythm is Erdős-deep if it has (exactly) one distance of multiplicity $i$, for each $i \in [1, k-1]$. Note that these multiplicities sum to $\sum_{i=1}^{k-1} i = \left\lfloor \frac{k}{2} \right\rfloor$, which is the cardinality of the geodesic distance multiset, and hence these distances are all the distances in the rhythm. Every geodesic distance is between 0 and $\lceil n/2 \rceil$. A rhythm is Winograd-deep if every two distances from $[1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor]$ have different multiplicity.

A shelling of an Erdős-deep rhythm $R$ is an ordering $s_1, s_2, \ldots, s_k$ of the onsets in $R$ such that $R - \{s_1, s_2, \ldots, s_i\}$ is an Erdős-deep rhythm for $i = 0, 1, \ldots, k$. (Every rhythm with at most two onsets is Erdős-deep.)

The evenness of rhythm $R$ is the sum of all inter-onset chordal distances in $R$; that is, $\sum_{0 \leq i < j \leq k-1} \overset{\rightarrow}{d}(r_i, r_j)$.

The clockwise distance sequence of $R$ is the circular sequence $(d_0, d_1, \ldots, d_{k-1})$ where $d_i = \overset{\rightarrow}{d}(r_i, r_{i+1})$ for all $i \in [0, k-1]$. Observe that each $d_i \in \mathbb{Z}^+$ and $\sum_i d_i = n$.

Observation 1. There is a one-to-one relationship between rhythms with $k$ onsets and timespan $n$ and circular sequences $(d_0, d_1, \ldots, d_{k-1})$ where each $d_i \in \mathbb{Z}^+$ and $\sum_i d_i = n$.

4. Even rhythms

In this section we first describe three algorithms that generate even rhythms. We then characterize rhythms with maximum evenness and show that, for given numbers of pulses and onsets, the three described algorithms generate the unique rhythm with maximum evenness. As mentioned in the introduction, the measure of evenness considered here is the pairwise sum of chordal distances.

The even rhythms characterized in this section were studied by Clough and Myerson [30,31] for the case where the numbers of pulses and onsets are relatively prime. This was subsequently expanded upon by Clough and Douthett [22]. We revisit these results and provide an additional connection to rhythms (and scales) that are obtained from the Euclidean algorithm. Most of these results are stated in [22]. However our proofs are new, and in many cases are much more streamlined.

4.1. Characterization

We first present three algorithms for computing a rhythm with $k$ onsets, timespan $n$, for any $k \leq n$, that possess large evenness.

The first algorithm is by Clough and Douthett [22]:
Algorithm. Clough-Douthett(k, n)

1. return \([\lfloor \frac{n}{k} \rfloor : i \in [0, k-1]]\).

Because \(k \leq n\), the rhythm output by Clough-Douthett(k, n) has \(k\) onsets as desired.

The second algorithm is a geometric heuristic implicit in the work of Clough and Douthett [22]:

Algorithm. SNAP(k, n)

1. Let \(D\) be a set of \(k\) evenly spaced points on \(C_n\) such that \(D \cap C_n = \emptyset\).
2. For each point \(x \in D\), let \(x'\) be the first point in \(C_n\) clockwise from \(x\).
3. \text{return} \(\{x' : x \in D\}\).

Because \(k \leq n\), the clockwise distance between consecutive points in \(D\) in the execution of SNAP(k, n) is at least that of consecutive points in \(C_n\). Thus, \(x' \neq y'\) for distinct \(x, y \in D\), so SNAP returns a rhythm with \(k\) onsets as desired.

The third algorithm is a recursive algorithm in the same mold as Euclid's algorithm for greatest common divisors. The algorithm uses the clockwise distance sequence notation described in the introduction. The resulting rhythm always defines the same necklace as the Euclidean rhythms from Section 2.3; that is, the only difference is a possible rotation.

Algorithm. Euclidean(k, n)

1. \text{if} \(k\) evenly divides \(n\) \text{ then return} \(\left(\frac{n}{k}, \frac{n}{k}, \ldots, \frac{n}{k}\right)\).
2. \(a \leftarrow n \mod k\).
3. \((x_1, x_2, \ldots, x_a) \leftarrow \text{Euclidean}(k, a)\).
4. \text{return} \(\left(\lfloor \frac{n}{k} \rfloor, \lfloor \frac{n}{k} \rfloor, \lfloor \frac{n}{k} \rfloor, \ldots, \lfloor \frac{n}{k} \rfloor, \lfloor \frac{n}{k} \rfloor \right)\).

As a simple example, consider \(k = 5\) and \(n = 13\). The sequence of calls to Euclidean(k, n) follows the same pattern as the Euclid algorithm for greatest common divisors from Section 2.1, except that it now stops one step earlier:

\(5, 13\), \(3, 5\), \(2, 3\), \(1, 2\). At the base of the recursion, we have Euclidean(1, 2) = \((2) = [\times \cdot \cdot]\). At the next level up, we obtain Euclidean(2, 3) = \((1, 2) = [\times \times \cdot]\). Next we obtain Euclidean(3, 5) = \((2, 1, 2) = [\times \times \times \cdot \cdot]\). Finally, we obtain Euclidean(5, 13) = \((2, 3, 3, 2, 3) = [\times \cdot \times \cdot \cdot \times \cdot \cdot \cdot \cdot \cdot]\). (For comparison, the Euclidean rhythm from Section 2.2 is \(E(5, 13) = (2, 3, 2, 3, 3)\), a rotation by 5.)

We now show that algorithm Euclidean(k, n) outputs a circular sequence of \(k\) integers that sum to \(n\) (which is thus the clockwise distance sequence of a rhythm with \(k\) onsets and timespan \(n\)). We proceed by induction on \(k\). If \(k\) evenly divides \(n\), then the claim clearly holds. Otherwise \(a (= n \mod k) > 0\), and by induction \(\sum_{i=1}^{a} x_i = k\). Thus the sequence that is output has \(k\) terms and sums to

\[
\frac{n}{k} + \frac{n}{k} \sum_{i=1}^{a} (x_i - 1) = a \left\lfloor \frac{n}{k} \right\rfloor + (k - a) \left\lfloor \frac{n}{k} \right\rfloor
\]

\[
= a \left(1 + \left\lfloor \frac{n}{k} \right\rfloor\right) + (k - a) \left\lfloor \frac{n}{k} \right\rfloor
\]

\[
= a + k \left\lfloor \frac{n}{k} \right\rfloor
\]

\[
= n.
\]

The following theorem is one of the main contributions of this paper.

Theorem 4.1.

Let \(n \geq k \geq 2\) be integers. The following are equivalent for a rhythm \(R = [r_0, r_1, \ldots, r_{k-1}]_n\) with \(k\) onsets and timespan \(n\):

(A) \(R\) has maximum evenness (sum of pairwise inter-onset chordal distances);
(B) \(R\) is a rotation of the Clough-Douthett(k, n) rhythm;
(C) \(R\) is a rotation of the SNAP(k, n) rhythm;
(D) \(R\) is a rotation of the Euclidean(k, n) rhythm; and

(\(\ast\)) for all \(i \in [1, k]\) and \(i \in [0, k - 1]\), the ordered pair \((r_i, r_{i+\ell})\) has clockwise distance \(\bar{d}(r_i, r_{i+\ell}) \in \{\left\lfloor \frac{n}{k} \right\rfloor, \left\lceil \frac{n}{k} \right\rceil\}\).

Moreover, up to a rotation, there is a unique rhythm that satisfies these conditions.
Note that the evenness of a rhythm equals the evenness of the same rhythm played backwards. Thus, if \( R \) is the unique rhythm with maximum evenness, then \( R \) is the same rhythm as \( R \) played backwards (up to a rotation).

The proof of Theorem 4.1 proceeds as follows. In Section 4.2 we prove that each of the three algorithms produces a rhythm that satisfies property (⋆). Then in Section 4.3 we prove that there is a unique rhythm that satisfies property (⋆). Thus the three algorithms produce the same rhythm, up to rotation. Finally in Section 4.4 we prove that the unique rhythm that satisfies property (⋆) maximizes evenness.

4.2. Properties of the algorithms

We now prove that each of the algorithms has property (⋆). Clough and Douthett [22] proved the following.

**Proof (B) ⇒ (⋆).** Say \( R = [r_0, r_1, \ldots, r_{k-1}]_n \) is the Clough-Douthett\((k, n)\) rhythm. Consider an ordered pair \((r_1, r_{1+\ell})\) of onsets in \( R \). Let \( p_1 = in \mod k \) and let \( p_\ell = \ell n \mod k \). By symmetry we can suppose that \( r_i \leq r_{i+\ell} \mod k \). Then the clockwise distance \( \bar{d}(r_1, r_{1+\ell}) \) is

\[
\left( \frac{(i+\ell)n}{k} \right) - \left( \frac{in}{k} \right) + \left( \frac{\ell n}{k} \right) + \left( \frac{p_\ell + p_i}{k} \right) - \left( \frac{in}{k} \right) = \left( \frac{\ell n}{k} \right) + \left( \frac{p_\ell + p_i}{k} \right),
\]

which is \([\frac{\ell n}{k}]\) or \([\frac{\ell n}{k}]\), because \([\frac{p_\ell + p_i}{k}] \in \{0, 1\} \).

A similar proof shows that the rhythm \([\frac{\ell n}{k}]: i \in [0, k-1]\) satisfies property (⋆). Observe that (⋆) is equivalent to the following property.

(⋆⋆) If \((d_0, d_1, \ldots, d_{k-1})\) is the clockwise distance sequence of \( R \), then for all \( \ell \in [1, k] \), the sum of any \( \ell \) consecutive elements in \((d_0, d_1, \ldots, d_{k-1})\) equals \([\frac{\ell n}{k}]\) or \([\frac{\ell n}{k}]\).

**Proof (C) ⇒ (⋆⋆).** Let \((d_0, d_1, \ldots, d_{k-1})\) be the clockwise distance sequence of the rhythm determined by Snap\((k, n)\). For the sake of contradiction, suppose that for some \( \ell \in [1, k] \), the sum of \( \ell \) consecutive elements in \((d_0, d_1, \ldots, d_{k-1})\) is greater than \([\frac{\ell n}{k}]\). The case in which the sum is less than \([\frac{\ell n}{k}]\) is analogous. We can assume that these \( \ell \) consecutive elements are \((d_0, d_1, \ldots, d_{\ell-1})\). Using the notation defined in the statement of the algorithm, let \( x_0, x_1, \ldots, x_\ell \) be the points in \( D \) such that \( \bar{d}(x_0, x_{\ell+1}) = d_0 \) for all \( i \in [0, \ell - 1] \). Thus \( \bar{d}(x_0, x_{\ell+1}) \geq [\frac{\ell n}{k}] + 1 \). Now \( \bar{d}(x_{\ell}, x_{\ell+1}) < 1 \). Thus \( \bar{d}(x_1, x_{\ell+1}) > [\frac{\ell n}{k}] \geq \frac{\ell n}{k} \), which implies that \( \bar{d}(x_1, x_{\ell+1}) > \frac{\ell n}{k} \). This contradicts the fact that the points in \( D \) were evenly spaced around \( C_n \) in the first step of the algorithm.

**Proof (D) ⇒ (⋆⋆).** We proceed by induction on \( k \). Let \( R = \text{Euclidean}(k, n) \). If \( k \) evenly divides \( n \), then \( R = \left\{ \frac{r}{k}, \frac{n}{k}, \ldots, \frac{\ell n}{k} \right\} \), which satisfies (D). Otherwise, let \( a = n \mod k \) and let \((x_1, x_2, \ldots, x_\ell) = \text{Euclidean}(k, a) \). By induction, for all \( \ell \in [1, a] \), the sum of any \( \ell \) consecutive elements in \((x_1, x_2, \ldots, x_\ell)\) equals \([\frac{\ell n}{k}]\) or \([\frac{\ell n}{k}]\). Let \( S \) be a sequence of \( m \) consecutive elements in \( R \). By construction, for some \( 1 \leq i \leq j \leq a \), and for some \( 0 \leq s \leq x_i - 1 \) and \( 0 \leq t \leq x_j - 1 \), we have

\[
S = \left( \begin{array}{cccc}
\frac{n}{k} & \cdots & \frac{n}{k} & \frac{n}{k} \\
\frac{n}{k} & \cdots & \frac{n}{k} & \frac{n}{k} \\
& \cdots & \frac{n}{k} & \frac{n}{k} \\
& & \cdots & \frac{n}{k} \\
\end{array} \right)_{i \leq j \leq a}
\]

It remains to prove that \([\frac{mn}{k}] \leq \sum S \leq [\frac{mn}{k}]\).

We first prove that \( \sum S \geq [\frac{mn}{k}] \). We can assume the worst case for \( \sum S \) to be minimal, which is when \( s = x_i - 1 \) and \( t = x_j - 1 \). Thus by induction,

\[
m + 1 = \sum_{a=i}^{j} x_a \leq \left( \frac{(j-i+1)k}{a} \right).
\]

Hence

\[
\frac{am}{k} = \frac{a \left( \frac{(j-i+1)k}{a} \right)}{a} = \left( \frac{(j-i+1)k}{a} \right) = \frac{a}{k} = j - i + 1 - \frac{1}{k}.
\]

Thus \([\frac{am}{k}] \leq j - i \) and

\[
\sum S = \left\lfloor \frac{m}{k} \right\rfloor + j - i \geq m \left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{am}{k} \right\rfloor \geq m \left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{am}{k} \right\rfloor = m \left( \frac{n}{k} + \frac{am}{k} \right) = \left( \frac{m}{k} \right) \left( \frac{n}{k} + a \right) = \left( \frac{mn}{k} \right).
\]
Now we prove that $\sum S \leq \lfloor \frac{mn}{k} \rfloor$. We can assume the worst case for $\sum S$ to be maximal, which is when $s = 0$ and $t = 0$. Thus by induction,

$$m - 1 = \sum_{\alpha = i+1}^{j} x_{\alpha} \geq \left\lfloor \frac{(j-i-1)k}{a} \right\rfloor.$$ 

Hence

$$am \equiv a \left\lfloor \frac{(j-i-1)k}{a} \right\rfloor + \frac{a}{k} \left( \frac{(j-i-1)k - a + 1}{a} \right) + \frac{a}{k} = j - i + 1 + \frac{1}{k}.$$ 

Thus $\lfloor \frac{am}{k} \rfloor \geq j - i$ and

$$\sum S = m \left\lfloor \frac{n}{k} \right\rfloor + j - i \leq m \left\lfloor \frac{n}{k} \right\rfloor + \frac{am}{k} = m \left\lfloor \frac{n}{k} \right\rfloor + \frac{am}{k} = m \left( \frac{k}{k} \frac{n}{k} + a \right) = \left\lfloor \frac{mn}{k} \right\rfloor.$$ 

\[4.3.\text{Uniqueness}\]

In this section we prove that there is a unique rhythm satisfying the conditions in Theorem 4.1. The following well-known number-theoretic lemmas will be useful. Two integers $x$ and $y$ are inverses modulo $m$ if $xy \equiv 1 \pmod{m}$.

**Lemma 4.2.** (See [82], page 55.) An integer $x$ has an inverse modulo $m$ if and only if $x$ and $m$ are relatively prime. Moreover, if $x$ has an inverse modulo $m$, then it has an inverse $y \in [1, m - 1]$.

**Lemma 4.3.** If $x$ and $m$ are relatively prime, then $ix \not\equiv jx \pmod{m}$ for all distinct $i, j \in [0, m - 1]$.

**Proof.** Suppose that $ix \equiv jx \pmod{m}$ for some $i, j \in [0, m - 1]$. By Lemma 4.2, $x$ has an inverse modulo $m$. Thus $i \equiv j \pmod{m}$, and $i = j$ because $i, j \in [0, m - 1]$. \[\square\]

**Lemma 4.4.** For all relatively prime integers $n$ and $k$ with $2 \leq k \leq n$, there is an integer $\ell \in [1, k - 1]$ such that:

(a) $\ell n \equiv 1 \pmod{k}$,
(b) $i \ell \not\equiv j \ell \pmod{k}$ for all distinct $i, j \in [0, k - 1]$, and
(c) $i \ell \not\equiv j \ell \pmod{n}$ for all distinct $i, j \in [0, k - 1]$.

**Proof.** By Lemma 4.2 with $x = n$ and $m = k$, $n$ has an inverse $\ell$ modulo $k$. This proves (a). Thus $k$ and $\ell$ are relatively prime by Lemma 4.2 with $x = \ell$ and $m = k$. Hence (b) follows from Lemma 4.3. Let $t = (\lfloor \frac{n}{k} \rfloor)$. Then $\ell n = kt + 1$. By Lemma 4.3 with $m = n$ and $x = t$ (and because $k \leq n$), to prove (c) it suffices to show that $t$ and $n$ are relatively prime. Let $g = \gcd(t, n)$. Thus $\ell \frac{n}{k} = k \ell t + \ell$. Because $\frac{n}{k}$ and $\frac{t}{k}$ are integers, $\frac{1}{k}$ is an integer and $g = 1$. This proves (c). \[\square\]

The following theorem is the main result of this section.

**Theorem 4.5.** For all integers $n$ and $k$ with $2 \leq k \leq n$, there is a unique rhythm with $k$ onsets and timespan $n$ that satisfies property (\star), up to a rotation.

**Proof.** Let $R = \{r_0, r_1, \ldots, r_{k-1}\}_n$ be a $k$-onset rhythm that satisfies (\star). Recall that the index of an onset is taken modulo $k$, and that the value of an onset is taken modulo $n$. That is, $r_i = x$ means that $r_{i \text{ mod } k} = x \text{ mod } n$.

Let $g = \gcd(k, n)$. We consider three cases for the value of $g$.

**Case 1.** $g = k$: Because $R$ satisfies property (\star) for $\ell = 1$, every ordered pair $(r_i, r_{i+1})$ has clockwise distance $\frac{\ell}{k}$. By a rotation of $R$ we can assume that $r_0 = 0$. Thus $r_i = \frac{in}{k}$ for all $i \in [0, k - 1]$. Hence $R$ is uniquely determined in this case.

**Case 2.** $g = 1$ (see Fig. 7): By Lemma 4.4(a), there is an integer $\ell \in [1, k - 1]$ such that $\ell n \equiv 1 \pmod{k}$. Thus $\ell n = (k - 1)\lfloor \frac{n}{k} \rfloor + \lfloor \frac{n}{k} \rfloor$. Hence, of the $k$ ordered pairs $(r_i, r_{i+\ell})$ of onsets, $k - 1$ have clockwise distance $\lfloor \frac{n}{k} \rfloor$ and one has clockwise distance $\lfloor \frac{n}{k} \rfloor$. By a rotation of $R$ we can assume that $r_0 = 0$ and $r_{k-\ell} = n - \lfloor \frac{n}{k} \rfloor$. Thus $r_{i\ell} = i\lfloor \frac{n}{k} \rfloor$ for all $i \in [0, k - 1]$; that is, $r_{i\ell \text{ mod } k} = (i\lfloor \frac{n}{k} \rfloor) \mod n$. By Lemma 4.4(b) and (c), this defines the $k$ distinct onsets of $R$. Hence $R$ is uniquely determined in this case.

**Case 3.** $g = 1$ (for $\ell = 1$).
Thus by Eq. (2),
\[ r_0 = 0 \quad \text{and} \quad r_{k-l} = r_4 = 6 \quad \text{the darker dots}. \]
Then as shown by the arrows, the positions of the other onsets are implied.

Fig. 7. Here we illustrate Case 2 with \( n = 12 \) and \( k = 7 \). Thus \( \ell = 3 \) because \( 3 \times 12 = 1 \mod 7 \). We have \( \left\lceil \frac{\ell n}{k} \right\rceil = 6 \) and \( \left\lfloor \frac{\ell n}{k} \right\rfloor = 5 \). By a rotation we can assume that \( r_0 = 0 \) and \( r_{k-l} = r_4 = 6 \) (the darker dots). As shown by the arrows, these onsets imply the positions of the next three onsets (medium or green dots), which in turn imply the positions of the final three onsets (the light or yellow dots). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

**Case 3.** Let \( k' = \frac{k}{g} \) and let \( n' = \frac{n}{g} \). Observe that both \( k' \) and \( n' \) are integers. Because \( R \) satisfies \((\ast)\) and \( \left\lceil \frac{\ell n}{k} \right\rceil = \left\lfloor \frac{k'n}{k'} \right\rfloor = n' \), we have \( d(r_i, r_{i+k'}) = n' \) for all \( i \in [0, k-1] \). Thus
\[ r_{ik'+j} = in' + r_j \quad \text{for all} \quad i \in [0, g-1] \quad \text{and} \quad j \in [0, n'-1]. \]

Now \( \gcd(n', k') = 1 \) by the maximality of \( g \). By Lemma 4.4(a), there is an integer \( \ell' \in [1, k'-1] \) such that \( \ell' n' \equiv 1 \mod k' \). Thus \( \ell' n' = (k'-1)(\left\lceil \frac{\ell n}{k} \right\rceil) + \left\lfloor \frac{\ell n}{k} \right\rfloor \), implying \( \ell' n = (k-g)\left\lceil \frac{\ell n}{k} \right\rceil + g\left\lfloor \frac{\ell n}{k} \right\rfloor \). Hence, of the \( k \) ordered pairs \((r_i, r_{i+k'})\) of onsets, \( k - g \) have clockwise distance \( \left\lceil \frac{\ell n}{k} \right\rceil \) and \( g \) have clockwise distance \( \left\lfloor \frac{\ell n}{k} \right\rfloor \). By a rotation of \( R \) we can assume that \( r_0 = 0 \) and \( r_{k'} = \left\lceil \frac{\ell n'}{k'} \right\rceil \). By Eq. (1) with \( j = 0 \) and \( j = \ell' \), we have
\[ r_{ik'} = in' \quad \text{and} \quad r_{ik'+\ell'} = in' + \left\lfloor \frac{\ell' n'}{k'} \right\rfloor \quad \text{for all} \quad i \in [0, g-1]. \]

This accounts for the \( g \) ordered pairs \((r_i, r_{i+k'})\) with clockwise distance \( \left\lceil \frac{\ell n}{k} \right\rceil \). The other \( k - g \) ordered pairs \((r_i, r_{i+k'})\) have clockwise distance \( \left\lfloor \frac{\ell n}{k} \right\rfloor \). Define
\[ L_0 = 0 \quad \text{and} \quad L_j = \left\lceil \frac{\ell' n'}{k'} \right\rceil + (j-1) \left\lfloor \frac{\ell' n'}{k'} \right\rfloor \quad \text{for all} \quad j \in [1, k'-1]. \]

Thus by Eq. (2),
\[ r_{ik'+j\ell'} = in' + L_j \quad \text{for all} \quad i \in [0, g-1] \quad \text{and} \quad j \in [0, k'-1]; \]
that is, \( r_{(ik'+j\ell')} \mod k = (in' + L_j) \mod n \).

To conclude that \( R \) is uniquely determined, we must show that over the range \( i \in [0, g-1] \) and \( j \in [0, k'-1] \), the numbers \( ik' + j\ell' \) are distinct modulo \( k \), and the numbers \( in' + L_j \) are distinct modulo \( n \).

First we show that the numbers \( ik' + j\ell' \) are distinct modulo \( k \). Suppose that
\[ ik' + j\ell' \equiv pk' + j\ell' \mod k \]
for some \( i, p \in [0, g-1] \) and \( j, q \in [0, k'-1] \). Because \( k = k' \cdot g \), we can write \((ik' + j\ell') \mod k \) as a multiple of \( k' \) plus a residue modulo \( k' \). In particular,
\[ (ik' + j\ell') \mod k = k'\left(i + \left\lceil \frac{j\ell'}{k'} \right\rceil \mod g\right) + (j\ell' \mod k'). \]
Thus Eq. (3) implies that
\[ k' \left( \left( i + \frac{L_j}{n} \right) \mod g \right) + \left( j \ell' \mod k' \right) = k' \left( \left( p + \frac{qL'}{k} \right) \mod g \right) + \left( q \ell' \mod k' \right). \]  
(4)

Hence \( j \ell' \equiv qL' \mod k' \). Thus \( j = q \) by Lemma 4.4(c). By substituting \( j = q \) into Eq. (4), it follows that \( i \equiv p \mod g \). Thus \( i = p \) because \( i, p \in [0, g - 1] \). This proves that the numbers \( i k' + j \ell' \) are distinct modulo \( k \).

Now we show that the numbers \( in' + L_j \) are distinct modulo \( n \). The proof is similar to the above proof that the numbers \( ik' + j \ell' \) are distinct modulo \( k \).

Suppose that
\[ in' + L_j \equiv pn' + L_q \pmod n \]  
(5)
for some \( i, p \in [0, g - 1] \) and \( j, q \in [0, k' - 1] \). Because \( n = n' \cdot g \), we can write \((in' + L_j) \mod n \) as a multiple of \( n' \) plus a residue modulo \( n' \). In particular,
\[ (in' + L_j) \mod n = n' \left( i + \left\lfloor \frac{L_j}{n'} \right\rfloor \right) \mod g + (L_j \mod n'). \]

Thus Eq. (5) implies that
\[ n' \left( i + \left\lfloor \frac{L_j}{n'} \right\rfloor \right) \mod g + (L_j \mod n') = n' \left( p + \left\lfloor \frac{L_q}{n'} \right\rfloor \right) \mod g + (L_q \mod n'). \]  
(6)

Hence \( L_j \equiv L_q \pmod {n'} \). We claim that \( j = q \). If \( j = 0 \) then \( L_j = 0 \), implying \( L_q = 0 \) and \( q = 0 \). Now assume that \( j, q \geq 1 \). In this case, \( L_j = j \left\lfloor \frac{L_j}{n'} \right\rfloor + 1 \) and \( L_q = q \left\lfloor \frac{L_q}{n'} \right\rfloor + 1 \). Thus
\[ j \left\lfloor \frac{L_j}{k'} \right\rfloor = q \left\lfloor \frac{L_q}{k'} \right\rfloor \mod k'. \]

Hence \( j = q \) by Lemma 4.4(c). By substituting \( j = q \) into Eq. (6), it follows that \( i \equiv p \pmod g \). Thus \( i = p \) because \( i, p \in [0, g - 1] \). This proves that the numbers \( in' + L_j \) are distinct modulo \( n \).

Therefore \( R \) is uniquely determined.  

We have shown that each of the three algorithms generates a rhythm with property (\( \ast \)), and that there is a unique rhythm with property (\( \ast \ast \)). Thus all of the algorithms produce the same rhythm, up to rotation. It remains to prove that this rhythm has maximum evenness.

4.4. Rhythms with maximum evenness

We start with a technical lemma. Let \( v, w \) be points at geodesic distance \( d \) on a circle \( C \). Obviously \( \overline{d}(v, w) \) is a function of \( d \), independent of \( v \) and \( w \). Let \( f(C, d) = \overline{d}(v, w) \).

**Lemma 4.6.** For all geodesic lengths \( x \leq d \) on a circle \( C \), we have \( f(C, x) + f(C, d - x) \leq 2 \cdot f(C, \frac{d}{2}) \), with equality only if \( d = 2x \).

**Proof.** We can assume that \( C \) is a unit circle. Consider the isosceles triangle formed by the center of \( C \) and a geodesic of length \( d \leq \pi \). We have \( \frac{1}{2} f(C, d) = \sin \frac{d}{2} \). Thus \( f(C, d) = 2 \sin \frac{d}{2} \). Thus our claim is equivalent to \( \sin x + \sin(d - x) \leq 2 \sin \frac{d}{2} \) for all \( 0 \leq x \leq d \). In the range \( 0 \leq x \leq d, \sin x \) is increasing, and \( \sin(d - x) \) is decreasing at the opposite rate. Thus \( \sin x + \sin(d - x) \) is maximized when \( x = d - x \). That is, when \( d = 2x \). The result follows.  

For a rhythm \( R = \{r_0, r_1, \ldots, r_{k-1}\}_R \), for each \( \ell \in [1, k] \), let \( S(R, \ell) \) be the sum of chordal distances taken over all ordered pairs \( (r_i, r_{i+\ell}) \) in \( R \). That is, let \( S(R, \ell) = \sum_{\ell=0}^{k-1} \overline{d}(r_i, r_{i+\ell}) \). Property (A) says that \( R \) maximizes \( \sum_{\ell=1}^k S(R, \ell) \). Before we characterize rhythms that maximize the sum of \( S(R, \ell) \), we first concentrate on rhythms that maximize \( S(R, \ell) \) for each particular value of \( \ell \). Let \( D(R, \ell) \) be the multiset of clockwise distances \( \{\overline{d}(r_i, r_{i+\ell}) : i \in [0, k - 1]\} \). Then \( S(R, \ell) \) is determined by \( D(R, \ell) \). In particular, \( S(R, \ell) = \sum \{f(C_n, d) : d \in D(R, \ell)\} \) (where \( \{f(C_n, d) : d \in D(R, \ell)\} \) is a multiset).

**Lemma 4.7.** Let \( 1 \leq \ell \leq k \leq n \) be integers. A \( k \)-onset rhythm \( R = \{r_0, r_1, \ldots, r_{k-1}\}_n \) maximizes \( S(R, \ell) \) if and only if \( \overline{d}(r_i, r_{i+\ell}) - \overline{d}(r_j, r_{j+\ell}) \leq 2 \) for all \( i, j \) in \( [0, k - 1] \).

**Proof.** Suppose that \( R = \{r_0, r_1, \ldots, r_{k-1}\}_n \) maximizes \( S(R, \ell) \). Let \( d_i = \overline{d}(r_i, r_{i+\ell}) \) for all \( i \in [0, k - 1] \). Suppose on the contrary that \( d_p > d_q + 2 \) for some \( p, q \in [0, k - 1] \). We can assume that \( q < p \), \( d_p = d_q + 2 \), and \( d_i = d_q + 1 \) for all \( i \in [q + 1, p - 1] \). Define \( r'_i = r_i + 1 \) for all \( i \in [q + 1, p] \), and define \( r'_i = r_i \) for all other \( i \). Let \( R' \) be the rhythm \( \{r'_0, r'_1, \ldots, r'_{k-1}\}_n \). Thus \( D(R, \ell) \subseteq D(R', \ell) \) and \( D(R', \ell) \setminus D(R, \ell) = \{d_p - 1, d_q + 1\} \). Now \( d_p - 1 = d_q + 1 = \frac{1}{2}(d_p + d_q) \). By Lemma 4.6, \( f(C_n, d_p) + f(C_n, d_q) < 2 \cdot f(C_n, \frac{1}{2}(d_p + d_q)) \). Thus \( S(R, \ell) < S(R', \ell) \), which contradicts the maximality of \( S(R, \ell) \).
For the converse, let $R$ be a rhythm such that $|\tilde{d}(r_i, r_{i+\ell}) - \tilde{d}(r_j, r_{j+\ell})| \leq 1$ for all $i, j \in [0, k-1]$. Suppose on the contrary that $R$ does not maximize $S(R, \ell)$. Thus some rhythm $T = (t_0, t_1, \ldots, t_{k-1})$ maximizes $S(T, \ell)$ and $T \neq R$. Hence $D(T, \ell) \neq D(R, \ell)$. Because $\sum D(R, \ell) = \sum D(T, \ell) (= \ell n)$, we have $\tilde{d}(t_i, t_{i+\ell}) - \tilde{d}(t_j, t_{j+\ell}) \geq 2$ for some $i, j \in [0, k-1]$. As we have already proved, this implies that $T$ does not maximize $S(T, \ell)$. This contradiction proves that $R$ maximizes $S(R, \ell)$. □

Because $\sum_{i=0}^{k-1} \tilde{d}(r_i, r_{i+\ell}) = \ell n$ for any rhythm with $k$ onsets and timespan $n$, Lemma 4.7 can be restated as follows.

Corollary 4.8. Let $1 \leq \ell \leq k \leq n$ be integers. A $k$-onset rhythm $R = \{r_0, r_1, \ldots, r_{k-1}\}$ maximizes $S(R, \ell)$ if and only if $\tilde{d}(r_i, r_{i+\ell}) \in \{\lceil\frac{\ell n}{T}\rceil, \lfloor\frac{\ell n}{T}\rfloor\}$ for all $i \in [0, k-1]$. □

Proof ($\ast$) ⇒ (A). If ($\ast$) holds for some rhythm $R$, then by Corollary 4.8, $R$ maximizes $S(R, \ell)$ for every $\ell$. Thus $R$ maximizes $\sum_\ell S(R, \ell)$. □

Proof (A) ⇒ ($\ast$). By Theorem 4.5, there is a unique rhythm $R$ that satisfies property (A). Let $R$ denote the unique rhythm that satisfies property ($\ast$). Suppose on the contrary that there is a rhythm $T = (t_0, t_1, \ldots, t_{k-1})$ with property (A) but $R \neq T$. Thus there exists an ordered pair $(t_i, t_{i+\ell})$ in $T$ with clockwise distance $\tilde{d}(t_i, t_{i+\ell}) \notin \{\lceil\frac{\ell n}{T}\rceil, \lfloor\frac{\ell n}{T}\rfloor\}$. By Corollary 4.8, $S(T, \ell) < S(R, \ell)$. Because $T$ has property (A), $\sum_{\ell=1}^{k} S(T, \ell) \geq \sum_{\ell=1}^{k} S(R, \ell)$. Thus for some $\ell'$ we have $S(T, \ell') > S(R, \ell')$. But this is a contradiction, because $S(R, \ell') \geq S(T, \ell')$ by Corollary 4.8. □

This completes the proof of Theorem 4.1. We now show that Theorem 4.1 can be generalized for other metrics that satisfy Lemma 4.6. To formalize this idea we introduce the following definition. A function $g : [0, \pi] \rightarrow \mathbb{R}^+$ is halving if for all geodesic lengths $x \leq d \leq \pi$ on the unit circle, $g(x) + g(d - x) \leq 2 \cdot g(\frac{d}{2})$, with equality only if $d = 2x$. For example, chord length is halving, but geodesic distance is not (because we have equality for all $x$). Observe that the proof of Lemma 4.7 and Corollary 4.8 depend on this property alone. Thus we have the following generalization of Theorem 4.1.

Theorem 4.9. Let $n \geq k \geq 2$ be integers. Let $g$ be a halving function. The following are equivalent for a rhythm $R = (r_0, r_1, \ldots, r_{k-1})$ with $n$ pulses and $k$ onsets:

(A) $R$ maximizes $\sum_{i=0}^{k-1} \sum_{j=i+1}^{k-1} g(\tilde{d}(r_i, r_j))$,
(B) $R$ is determined by the Clough-Douthett($k, n$) algorithm,
(C) $R$ is determined by the Snap($k, n$) algorithm,
(D) $R$ is determined by the Euclidean($k, n$) algorithm,

($\ast$) for all $\ell \in [1, k]$ and $i \in [0, k-1]$, the ordered pair $(r_i, r_{i+\ell})$ has clockwise distance $d(r_i, r_{i+\ell}) \in \{\lceil\frac{\ell n}{T}\rceil, \lfloor\frac{\ell n}{T}\rfloor\}$.

Moreover, up to a rotation, there is a unique rhythm that satisfies these conditions.

5. Deep rhythms

Recall that a rhythm is Winograd-deep if every geodesic distance $1, 2, \ldots, \lfloor n/2 \rfloor$ has a unique multiplicity; it is Erdős-deep if the multiplicity of every geodesic distance defined by pairs of onsets in unique. Winograd [94], and independently Clough et al. [23], characterize all Winograd-deep scales; up to rotation, they are the scales that can be generated by the first $\lfloor n/2 \rfloor$ or $\lfloor n/2 \rfloor + 1$ multiples (modulo $n$) of a value that is relatively prime to $n$, plus one exceptional scale $\{0, 1, 2, 4\}_6$. In this section, we prove a similar (but more general) characterization of Erdős-deep rhythms: up to rotation and scaling, they are the rhythms generated as the first $k$ multiples (modulo $n$) of a value that is relatively prime to $n$, plus the same exceptional rhythm $\{0, 1, 2, 4\}_6$. The key difference is that the number of onsets $k$ is now a free parameter, instead of being forced to be either $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil + 1$. Our proof follows Winograd’s, but differs in one case (the second case of Theorem 5.3).

We later prove that every Erdős-deep rhythm has a shelling and that maximally even rhythms with $n$ and $k$ relatively prime are Erdős-deep.

5.1. Characterization of deep rhythms

Our characterization of Erdős-deep rhythms is in terms of two families of rhythms. The main rhythm family consists of the generated rhythms $D_{k, n, m} = \{i \mod n : i = 0, 1, \ldots, k - 1\}_n$ of timespan $n$, for certain values of $k$, $n$, and $m$. The one exceptional rhythm is $F = \{0, 1, 2, 4\}_6$ of timespan 6.

Fact 5.1. $F$ is Erdős-deep.
Lemma 5.2. If $k \leq \lfloor n/2 \rfloor + 1$ and $m$ and $n$ are relatively prime, then $D_{k,n,m}$ is Erdős-deep.

Proof. The multiset of clockwise distances in $D_{k,n,m}$ is $(\{jm - im\} \mod n : i < j) = (\{j - i\} \mod n : i < j)$. There are $k - p$ choices of $i$ and $j$ such that $j - i = p$, so there are exactly $p$ occurrences of the clockwise distance $(pm) \mod n$ in the multiset. Each of these clockwise distances corresponds to a geodesic distance—either $(pm) \mod n$ or $(-pm) \mod n$, whichever is smaller (at most $n/2$). We claim that these geodesic distances are all distinct. Then the multiplicity of each geodesic distance $\pm pm \mod n$ is exactly $p$, establishing that the rhythm is Erdős-deep.

For two geodesic distances to be equal, we must have $\pm pm = \pm qm \mod n$ for some (possibly different) choices for the ± symbols, and for some $p \neq q$. By (possibly) multiplying both sides by $-1$, we obtain two cases: (1) $pm = qm \mod n$ and (2) $pm = -qm \mod n$. Because $m$ is relatively prime to $n$, by Lemma 4.2, $m$ has a multiplicative inverse modulo $n$. Multiplying both sides of the congruence by this inverse, we obtain $(1) p = q \mod n$ and $(2) p = -q \mod n$. Because $0 \leq i < j < k \leq \lfloor n/2 \rfloor + 1$, we have $0 \leq p = j - i < \lfloor n/2 \rfloor + 1$, and similarly for $q$: $0 \leq p, q \leq \lfloor n/2 \rfloor$. Thus, the first case of $p = q \mod n$ can happen only when $p = q$, and the second case of $p + q = 0 \mod n$ can happen only when $p = q = 0$ or when $p = q = n/2$. Either case contradicts that $p \neq q$. Therefore the geodesic distances arising from different values of $p$ are indeed distinct, proving the lemma. □

We now state and prove our characterization of Erdős-deep rhythms, which is up to rotation and scaling. Rotation preserves the geodesic distance multiset and therefore Erdős-deepness (and Winograd-deepness). Scaling maps each geodesic distance $d$ to $ad$, and thus preserves multiplicities and therefore Erdős-deepness (but not Winograd-deepness). Note that the rhythm $D_{k,n,m}$ is a rotation by $-m(k-1)$ mod $n$ of the rhythm $D_{k,n,n-m}$; to avoid this duplication we restrict $m$ to be equal to at most $\lfloor n/2 \rfloor$.

Theorem 5.3. A rhythm is Erdős-deep if and only if it is a rotation of a scaling of either the rhythm $F$ or the rhythm $D_{k,n,m}$ for some $k, n, m$ with $k \leq \lfloor n/2 \rfloor + 1$, $1 \leq m \leq \lfloor n/2 \rfloor$, and $m$ and $n$ are relatively prime.

Proof. Because a rotation of a scaling of an Erdős-deep rhythm is Erdős-deep, the “if” direction of the theorem follows from Fact 5.1 and Lemma 5.2.

Consider an Erdős-deep rhythm $R$ with $k$ onsets. By the definition of Erdős-deepness, $R$ has one nonzero geodesic distance with multiplicity $i$ for each $i = 1, 2, \ldots, k - 1$. Let $m$ be the geodesic distance with multiplicity $k - 1$. Because $m$ is a geodesic distance, $1 \leq m \leq \lfloor n/2 \rfloor$. Also, $k \leq \lfloor n/2 \rfloor + 1$ (for any Erdős-deep rhythm $R$), because all nonzero geodesic distances are between 1 and $\lfloor n/2 \rfloor$ and therefore at most $\lfloor n/2 \rfloor$ nonzero geodesic distances occur. Thus $k$ and $m$ are suitable parameter choices for $D_{k,n,m}$.

Consider the graph $G_m = (R, E_m)$ with vertices corresponding to onsets in $R$ and with an edge between two onsets of geodesic distance $m$. By the definition of geodesic distance, every vertex $i$ in $G_m$ has degree at most 2: the only onsets at geodesic distance exactly $m$ from $i$ are $(i - m) \mod n$ and $(i + m) \mod n$. Thus, the graph $G_m$ is a disjoint union of paths and cycles. The number of edges in $G_m$ is the multiplicity of $m$, which we supposed was $k - 1$, which is 1 less than the number of vertices in $G_m$. Thus, the graph $G_m$ consists of exactly one path and any number of cycles.

The cycles of $G_m$ have a special structure because they correspond to subgroups generated by single elements in the cyclic group $(\mathbb{Z}/(n), +)$. Namely, the onsets corresponding to vertices of a cycle in $G_m$ form a regular $(n/a)$-gon, with a geodesic distance of $a = \gcd(m,n)$ between consecutive onsets. (a is called the index of the subgroup generated by $m$.) In particular, every cycle in $G_m$ has the same length $r = n/a$. Because $G_m$ is a simple graph, every cycle must have at least 3 vertices, so $r \geq 3$.

The proof partitions into four cases depending on the length of the path and on how many cycles the graph $G_m$ has. The first two cases will turn out to be impossible; the third case will lead to a rotation of a scaling of rhythm $F$; and the fourth case will lead to a rotation of a scaling of rhythm $D_{k,n,m}$.

First suppose that the graph $G_m$ consists of a path of length at least 1 and at least one cycle. We show that this case is impossible because the rhythm $R$ can have no geodesic distance with multiplicity 1. Suppose that there is a geodesic distance with multiplicity 1, say between onsets $i_1$ and $i_2$. If $i$ is a vertex of a cycle, then both $(i + m) \mod n$ and $(i - m) \mod n$ are onsets in $R$. If $i$ is a vertex of the path, then one or two of these are onsets in $R$, with the case of one occurring only at the endpoints of the path. If $(i_1 + m) \mod n$ and $(i_2 + m) \mod n$ were both onsets in $R$, or $(i_1 - m) \mod n$ and $(i_2 - m) \mod n$ were both onsets in $R$, then we would have another occurrence of the geodesic distance between $i_1$ and $i_2$, contradicting that this geodesic distance has multiplicity 1. Thus, $i_1$ and $i_2$ must be opposite endpoints of the path. If the path has length $\ell$, then the clockwise distance between $i_1$ and $i_2$ is $(\ell m) \mod n$. This clockwise distance (and hence the corresponding geodesic distance) appears in every cycle, of which there is at least one, so the geodesic distance has multiplicity more than 1, a contradiction. Therefore this case is impossible.

Second suppose that the graph $G_m$ consists of a path of length 0 and at least two cycles. We show that this case is impossible because the rhythm $R$ has two geodesic distances with the same multiplicity. Pick any two cycles $C$ and $C'$, and let $d$ be the smallest positive clockwise distance from a vertex of $C$ to a vertex of $C'$. Thus $i$ is a vertex of $C$ if and only if $(i + d) \mod n$ is a vertex of $C'$. Because the cycles are disjoint, $d < a$. Because $r \geq 3$, $d < n/3$, so clockwise distances of $d$ are also geodesic distances of $d$. The number of occurrences of geodesic distance $d$ between a vertex of $C$ and a vertex of
C′ is either r or 2r, the case of 2r arising when \(d = a/2\) (that is, C′ is a “half-rotation” of C). The number of occurrences of geodesic distance \(d′ = \min(d + m, n - (d + m))\) is the same—either r or 2r, in the same cases. (Note that \(d < a \leq n - m\), so \(d + m < n\), so the definition of C′ correctly captures a geodesic distance modulo n.) The same is true of geodesic distance \(d'' = \min(d - m, n - (d - m))\). If other pairs of cycles have the same smallest positive clockwise distance d, then the number of occurrences of d, d′, and d″ between those cycles are also equal. Because the cycles are disjoint, geodesic distance d and thus \(d + m\) and \(d - m\) cannot be \((pm) \mod n\) for any p, so these geodesic distances cannot occur between two vertices of the same cycle. Finally, the sole vertex \(x\) of the path has geodesic distance d to onset i (which must be a vertex of some cycle) if and only if x has geodesic distance d′ to onset \((i + m) \mod n\) (which must be a vertex of the same cycle) if and only if x has geodesic distance d′′ to onset \((i - m) \mod n\) (which also must be a vertex of the same cycle). Therefore the multiplicities of geodesic distances d, d′, and d″ must be equal. Because R is Erdős-deep, we must have \(d = d′ = d″\). To have \(d = d′\), either \(d = d + m\) or \(d = n - (d + m)\), but the first case is impossible because \(d > 0\) by nonoverlap of cycles, so \(2d = m = n\). Similarly, to have \(d = d″\), we must have \(2d - m = n\). Subtracting these two equations, we obtain that \(2m = 0\), contradicting that \(m > 0\). Therefore this case is also impossible.

Third suppose that the graph \(G_m\) consists of a path of length 0 and exactly one cycle. We show that this case forces R to be a rotation of a scaling of a rhythm F because otherwise two geodesic distances m and \(m′\) have the same multiplicity. The number of occurrences of geodesic distance m in the cycle is precisely the length r of the cycle. Similarly, the number of occurrences of geodesic distance \(m′ = \min(2m, n - 2m)\) in the cycle is r. The sole vertex \(x\) on the path cannot have geodesic distance m or \(m′\) to any other onset (a vertex of the cycle) because then x would then be on the cycle. Therefore the multiplicities of geodesic distances m and \(m′\) must be equal. Because R is Erdős-deep, m must equal \(m′\), which implies that either \(m = 2m\) or \(m = n - 2m\). The first case is impossible because \(m > 0\). In the second case, \(3m = n\), that is, \(m = \frac{3n}{2}\).

Therefore, the cycle has r = 3 vertices, say at \(Δ, Δ + \frac{1}{2}n, Δ + \frac{3}{2}n\). The fourth and final onset \(x\) must be midway between two of these three onsets, because otherwise its geodesic distance to the three vertices are all distinct and therefore unique. No matter where \(x\) is so placed, the rhythm R is a rotation by \(Δ + c\frac{1}{2}n\) (for some \(c \in \{0, 1, 2\}\)) of a scaling by \(n/6\) of the rhythm F.

Finally suppose that \(G_m\) has no cycles, and consists solely of a path. We show that this case forces R to be a rotation of a scaling of a rhythm \(D_{k,n,m}\) with \(1 \leq m' \leq \lfloor n/2\rfloor\) and with \(m'\) and \(n'\) relatively prime. Let i be the onset such that \((i - m) \mod n\) is not an onset (the “beginning” vertex of the path). Consider rotating R by \(-i\) so that 0 is an onset in the resulting rhythm R − i. The vertices of the path in \(R - i\) form a subset of the subgroup of the cyclic group \((\mathbb{Z}/n, +)\) generated by the element m. Therefore the rhythm R − i = \(D_{k,n,m}\) = \(\{(im) \mod n: i = 0, 1, \ldots, k - 1\}\) n. The rhythm \(D_{k,n,a,m/a}\) has an appropriate value for the third argument: m/a and n/a are relatively prime (a = \(gcd(m,n)\)) and \(1 \leq m/a \leq \lfloor n/2\rfloor / a \leq \lfloor (n/a)/2\rfloor\). Also, \(k \leq \lfloor (n/a)/2\rfloor + 1\) because the only occurring geodesic distances are multiples of \(a\) and therefore the number \(k - 1\) of distinct geodesic distances is at most \(\lfloor (n/a)/2\rfloor\). Therefore R is a rotation by i of a scaling by \(a\) of \(D_{k,n,a,m/a}\) with appropriate values of the arguments.

**Corollary 5.4.** A rhythm is Erdős-deep if and only if it is a rotation of a scaling of the rhythm F or it is a rotation of a rhythm \(D_{k,n,m}\) for some \(k, n, m\) satisfying \(k \leq \lfloor n/2g\rfloor + 1\) where \(g = gcd(m, n)\).

**Proof.** First we show that any Erdős-deep rhythm has one of the two forms in the corollary. By Theorem 5.3, there are two flavors of Erdős-deep rhythms, and the corollary directly handles rotations of scalings of F. Thus it suffices to consider a rhythm R that is a rotation by \(\Delta\) of a scaling of \(\alpha\) of \(D_{k,n,m}\), where \(k \leq \lfloor n/2\rfloor + 1\), \(1 \leq m \leq \lfloor n/2\rfloor\), and m and n are relatively prime. Equivalently, R is a rotation by \(\Delta\) of \(D_{k,n,m'}\) where \(n' = \alpha n\) and \(m' = \alpha m\). Now \(g = gcd(n', m') = \alpha\), so \(n'/g = n\). Hence, \(k \leq \lfloor n'/g\rfloor + 1\) as desired. Thus we have rewritten R in the desired form.

It remains to show that every rhythm in one of the two forms in the corollary is Erdős-deep. Again, rotations of scalings of F are handled directly by Theorem 5.3. So consider a rotation of \(D_{k,n,m}\) where \(k \leq \lfloor n/2g\rfloor + 1\). The value of m matters only modulo n, so we assume that \(0 \leq m \leq n - 1\).

First we show that, if \([n/2] + 1 \leq m \leq n - 1\), then \(D_{k,n,m}\) can be rewritten as a rotation of the rhythm \(D_{k,n,m'}\) where \(m' = n - m \leq \lfloor n/2\rfloor\). By reversing the order in which we list the onsets in \(D_{k,n,m}\) = \(\{i \mod n: i = 0, 1, \ldots, k - 1\}\), we can write \(D_{k,n,m} = \{(k - 1 - i) \mod n: i = 0, 1, \ldots, k - 1\}\). Now consider rotating the rhythm \(D_{k,n,n-m}\) by \((k - 1)m\). We obtain the rhythm \([i (n - m) + (k - 1) m] \mod n: i = 0, 1, \ldots, k - 1\] = \(\{(k - 1 - i) \mod n: i = 0, 1, \ldots, k - 1\}\) as desired.

Thus it suffices to consider rotations of \(D_{k,n,m}\) where \(1 \leq m \leq \lfloor n/2\rfloor\) and \(k \leq \lfloor n/2g\rfloor\). The rhythm \(D_{k,n,m'}\), where \(n' = n/g\) and \(m' = m/g\), is Erdős-deep by Theorem 5.3 because \(n'\) and \(m'\) are relatively prime, \(k \leq \lfloor n'/g\rfloor + 1\), and \(1 \leq m' \leq \lfloor n'/g\rfloor\). But \(D_{k,n,m}\) is the scaling of \(D_{k,n,m'}\) by the integer \(g\), so \(D_{k,n,m}\) is also Erdős-deep.

An interesting consequence of this characterization is the following:

**Corollary 5.5.** Every Erdős-deep rhythm has a shelling.

**Proof.** If the Erdős-deep rhythm is \(D_{k,n,m}\), we can remove the last onset from the path, resulting in \(D_{k-1,n,m}\), and repeat until we obtain the empty rhythm \(D_{0,n,m}\). At all times, \(k\) remains at most \([n/2] + 1\) (assuming it was originally) and \(m\)
remains between 1 and \([n/2]\) and relatively prime to \(n\). On the other hand, \(F = \{0, 1, 2, 4\}_6\) has the shelling 4, 2, 1, 0 because \(0, 1, 2, 4\)_6 is Erdős-deep. □

We can generalize this characterization of Erdős-deep rhythms to the continuous case where \(n\) is an arbitrary real number, and onsets can be at arbitrary (not necessarily integer) points along the circle. We will call such rhythms continuous rhythms. In this case we have two kinds of rhythms. First, if \(m\) and \(n\) are rational multiples of each other, we can scale the rhythm by some rational \(p\) such that \(pm\) and \(pn\) are integers, and apply Theorem 5.3 using \(pn\) and \(pm\) to characterize all deep rhythms where \(m\) is a rational multiple of \(n\). Second, if \(m\) and \(n\) are irrational multiples of each other, we can show that every \(D_{k,n,m}\) is Erdős-deep. The complete characterization of continuous Erdős-deep rhythms is as follows:

**Theorem 5.6.** A continuous rhythm is Erdős-deep if and only if it is a rotation of a scaling of \(D_{k,n,m}\) with \(k \leq [n/2] + 1\), \(0 < m \leq n/2\), and where \(m\) and \(n\) are either (1) irrational multiples of each other, or (2) rational multiples such that for some rational \(p\), integers \(pm\) and \(pn\) are relatively prime.

**Proof.** To prove the “if” direction, we show that all geodesic distances defined by \(D_{k,n,m}\) are distinct; hence we need to prove that the multiplicity of each geodesic distance \((\pm pm)\mod n\) is exactly 1. First assume that \(m\) and \(n\) are irrational multiples of each other, i.e., there is no rational number that divides both \(m\) and \(n\). Suppose two geodesic distances \((\pm pm)\mod n\) for some (possibly different) choices for the \(\pm\) symbols, and for some \(p \neq q\). Then we can write \(\pm pm = \pm qm + r\) for some integer \(r\). This in turn implies that \(m = \frac{r}{\pm pq}n\), which contradicts the fact that \(m\) and \(n\) are irrational multiples of each other. Therefore, when \(m\) and \(n\) are irrational multiples of each other, the geodesic distances arising from different values of \(p\) are distinct, proving that \(D_{k,n,m}\) is Erdős-deep.

If \(m\) and \(n\) are rational multiples of each other, then so are each of the geodesic distances \(2m, 3m, \ldots, (k-1)m\mod n\) with \(n\). In this case, there exists a rational \(p\) such that \(pm\) and \(pn\) are both integers. We can now apply Theorem 5.3 using \(pn\) and \(pm\), and generate all deep rhythms where \(m\) is a rational multiple of \(n\).

For the “only if” direction, consider a continuous Erdős-deep rhythm \(R\) with \(k\) onsets and period \(n\), and with some geodesic distance \(m\) having multiplicity \(k - 1\). Consider the graph \(G_m = (R, E_m)\) as defined in the proof of Theorem 5.3 (with vertices corresponding to onsets in \(R\) and with an edge between two onsets of geodesic distance \(m\)). If \(m\) and \(n\) are rational multiples of each other, then we can scale \(R\) by some rational \(p\) and apply Theorem 5.3 to show that \(R\) is a scaling by \(1/p\) of \(D_{k,pn,pm}\) where \(pm\) and \(pn\) are relatively prime integers and \(1 \leq pm \leq [pn/2]\), so \(0 < m \leq n/2\).

If \(m\) and \(n\) are irrational multiples of each other, then there is no rational number \(r\) such that \(n = rm\). This means that \(G_m\) cannot contain a cycle, so consists of a single path of length \(k - 1\). As in the proof of Theorem 5.3, we can rotate \(R\) by \(-i\), so that \(0\) is an onset in the resulting rhythm \(R - i\). The vertices of the path in \(R - i\) form a subset of the subgroup of the cyclic group \((\mathbb{Z}/n, +)\) generated by the element \(m\). Therefore the rhythm \(R - i = D_{k,n,m} = (\{im\mod n: i = 0, 1, \ldots, k-1\}_n\) where \(m\) and \(n\) are irrational multiples of each other and \(0 < m \leq n/2\). □

### 5.2. Connection between deep and even rhythms

A connection between maximally even scales and Winograd-deep scales is shown by Clough et al. [23]. They define a **diatonic scale** to be a maximally even scale with \(k = (n + 2)/2\) and \(n\) a multiple of 4. They show that diatonic scales are Winograd-deep. We now prove a similar result for Erdős-deep rhythms.

**Lemma 5.7.** A rhythm \(R\) of maximum evenness satisfying \(k \leq [n/2] + 1\) is Erdős-deep if and only if \(k\) and \(n\) are relatively prime.

**Proof.** Recall that by property (⋆) one of the unique characterizations of an even rhythm of maximum evenness can be stated as follows. For all \(1 \leq \ell \leq k\), and for every ordered pair \((r_i, r_{i+\ell})\) of onsets in \(R\), the clockwise distance \(\overrightarrow{d}(r_i, r_{i+\ell}) \in \{[\ell/n], [\ell/2]\}\).

For the case in which \(k\) and \(n\) are relatively prime, by Lemma 4.2, there exists a value \(\ell < k\) such that \(\ell n \equiv 1\mod k\). Thus we can write \(\ell n = k \lfloor \ell n/k \rfloor + 1\). Let \(m = \lfloor \ell n/k \rfloor\). Now consider the set \(\{im \mod n: i = 0, 1, \ldots, k-1\}_n\). By Lemma 4.4(c), we get \(k\) distinct values, so \(R\) can be realized as \(D_{k,n,m} = \{im \mod n: i = 0, 1, \ldots, k-1\}_n\). Thus, by Lemma 5.2, \(R\) is Erdős-deep.

Observe that \(F = \{0, 1, 2, 4\}_6\) does not maximize evenness because \(\overrightarrow{d}(0, 2) = 2\) and \(\overrightarrow{d}(2, 0) = 4\), yet \(\ell = 2\). Hence, any rhythm that maximizes evenness and that is deep must also be generated.

Now consider the case in which \(n\) and \(k\) are not relatively prime. We show that the assumption that \(k\) is deep leads to a contradiction. Thus, assuming that \(k\) is deep implies that there is a value \(m\) such that \(R\) can be realized as \(D_{k,n,m} = \{im \mod n: i = 0, 1, \ldots, k-1\}_n\). This in turn implies that there exists an integer \(\ell\) such that \(km = \ell n + 1\), that is, \(\ell n \equiv 1\mod k\). However, for this to happen, \(n\) and \(k\) must be relatively prime, a contradiction.

Thus we have shown that \(R\) is Erdős-deep if and only if \(k\) and \(n\) are relatively prime. □
Acknowledgements

This work was initiated at the 20th Bellairs Winter Workshop on Computational Geometry held January 28–February 4, 2005 at Bellairs Research Institute in Barbados. We thank the other participants of that workshop—Greg Aloupis, David Bremner, Justin Colannino, Mirela Damian, Vida Dujmović, Jeff Erickson, Ferran Hurtado, John Iacono, Danny Krizanc, Stefan Langerman, Erin McLeish, Pat Morin, Mark Overmars, Suneeta Ramaswami, Diane Souvaine, Ileana Streinu, Remco Veltcamp, and Sue Whitesides—for helpful discussions and contributing to a fun and creative atmosphere.

Thanks to Simha Arom for providing us with a copy of his paper on the classification of *aksak* rhythms [4], which provided several examples of Euclidean rhythms of which we were not aware. Thanks to Jeff Erickson for bringing to our attention the wonderful paper by Mitchell Harris and Ed Reingold [51] on digital line drawing, leap year calculations, and Euclid’s algorithm. Finally, thanks to Marcia Ascher for pointing out that the leap-year pattern of the Jewish calendar is a Euclidean rhythmic necklace.

Finally, we would like to thank the two unknown referees for useful comments.

Appendix A. Euclidean rhythms in traditional world music

Below is a list of Euclidean rhythms that can be found in traditional world music. We restrict out attention to rhythms where $k$ and $n$ are relatively prime.

$E(2, 3) = [\times \times \cdot] = (12)$ is a common Afro–Cuban drum pattern when started on the second onset as in $[\times \times]$. For example, it is the conga rhythm of the (6/8)-time *Swing Tumbao* [56]. It is common in Latin American music, as for example in the *Cucu* [92], and the *coros de clave* [76]. It is common in Arabic music, as for example in the *Al Tā‘er* rhythm of Nubia [48]. It is also a rhythmic pattern of the Drum Dance of the Slavey Indians of Northern Canada [6].

$E(2, 5) = [\times \times \times \times \cdot] = (23)$ is a rhythm found in Greece, Namibia, Rwanda and Central Africa [4]. It is also a 13th century Persian rhythm called *Khajīf-e-ramal* [96], as well as the rhythm of the Macedonian dance *Makedonka* [79]. Tchaikovsky used it as the metric pattern in the second movement of his Symphony No. 6 [55]. Started on the second onset as in $[\times \times \times \times \cdot]$ it is a rhythm found in Central Africa, Bulgaria, Turkey, Turkestan and Norway [4]. It is also the metric pattern of Dave Brubeck’s *Take Five*.

$E(4, 5) = [\times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \time...
E(5, 16) = [x·x···x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x] = (33334) is the Bossa-Nova rhythm necklace of Brazil. The actual Bossa-Nova rhythm usually starts on the third onset as follows: [x·x···x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x] [85]. However, other starting places are also documented in world music practices, such as [x·x···x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x] [13].

E(6, 7) = [x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·] = (1111112) is the Pontákos rhythm of Greece when started on the sixth (last) onset [48].

E(6, 13) = [x·x···x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x] = (2222223) is the rhythm of the Macedonian dance Mama Cone pita [79]. Started on the third onset, it is the rhythm of the Macedonian dance Postupano Oro [79], as well as the Krivo Plovdivsko Horo of Bulgaria [75].

E(7, 8) = [x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·x·